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VARIATIONAL PRINCIPLES FOR LINEAR COUPLED THERMOELASTICITY WITH MICROSTRUCTURE

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Abstract—Variational principles are developed for the initial-boundary value problem of fully coupled linear thermoelasticity for inhomogeneous, anisotropic materials with microstructure. Alternative characterizations of the solution to the mixed problem are obtained using operational methods. The variational principles are formulated in such a fashion as to allow their almost immediate adaptation to all available theories of structured or generalized continua.

1. INTRODUCTION

IN THIS paper variational principles are established for linear coupled thermoelasticity including effects of material microstructure. Insofar as possible the formulation follows the systematic treatment employed by Gurtin [1] in his development of variational principles for linear elastodynamics. Gurtin's principles [1] fully allow for prescription of initial displacement and velocity fields, in contrast to the classical variational principles (Hamilton's principle) which allow specification of the displacement field at the initial instant but assume knowledge of the displacement field at a later instant and hence do not allow for initial conditions on velocities.

In treating thermoelasticity with microstructure we wish to specify initial conditions on the temperature field, the displacement and velocity fields associated with material particles, and the displacement and velocity fields associated with the microstructure deformation. Thus, guided by Gurtin [1, 2], we utilize the operational calculus of Mikusinski [3] in deriving a boundary value problem which is equivalent to the initial-boundary value problem of fully coupled linear thermoelasticity with microstructure. The field equations of this equivalent boundary value problem involve convolutions and implicitly contain the initial conditions. This general approach stems from work by Ignaczak, see [1, 4].

Classical variational principles for linear thermoelasticity are discussed by Nickell and Sackman [5], who recently extended Gurtin's [1] work to obtain variational principles for linear coupled thermoelasticity which take into account initial temperature, displacement and velocity distributions. The present paper extends the work of Nickell and Sackman to allow full consideration of material microstructure effects.

The theory of materials with microstructure now assumes a rather eclectic form, with contributions of a permanent character due to a number of researchers. Since our linear formulation is derived from such works, it is appropriate to devote this paragraph to a brief discussion of the development of modern theories of generalized continua and materials with microstructure. Dates given in the text refer to when manuscripts were received for publication and provide striking illustration of the high level of activity in this field. Ericksen and Truesdell [6] (19 March 1958) generalized and formalized the notion of an oriented body originally proposed in the seminal papers of Duhem and the Cosserats.

Ericksen and Truesdell considered a body to be composed of material particles and directors, with the directors capable of rotating and stretching independently of the deformation field associated with material particles. The main thrust of their work dealt with the kinematics of such media and they gave an exact description of stress and strain in rods and shells. However, their contribution transcends any listing of results obtained since they exposed to researchers a rich and theretofore somewhat neglected subject. Also in 1958, Gunther [7] discussed the connection between the Cosserat theory and the continuum theory of dislocations and gave a principle of virtual work for materials having independent particle deformation and rigid microstructure rotation fields. We omit mention of a number of important contributions made from 1959 to 1963, since they are discussed in [8–17]. Mindlin [8] (6 November 1963) developed a linear theory of elasticity with microstructure based on the notion of a unit cell which was used to represent properties of a crystal lattice and bears a variety of other interpretations, e.g. as a grain of a granular material. He exhibited the twelve displacement equations of motion and considered several approximate forms to examine micro-vibrations and wave propagation. Koiter [9] (16 November 1963) proposed the use of couple stress theory in explaining elastic fatigue. Eringen and Suhubi [10] (6 December 1963) gave a nonlinear theory of elasticity with microstructure which contained a laudable treatment of microstructure inertia and a derivation of the microstructure continuity equation. In a companion article Eringen [11] (11 December 1963) treated microfluids. Green and Rivlin [12] (23 December 1963) used the concept of force and stress multipoles, discussed in the treatise of Truesdell and Toupin [18], and considered velocity gradients of various orders in developing a theory of generalized continua. They showed that the equations of motion may be deduced from an energy postulate by use of invariance conditions under superposed rigid body motions and, for the case of generalized elasticity, made effective use of the entropy production inequality postulate in deriving expressions for multipolar stresses in terms of derivatives of the Helmholtz free energy function with respect to various strain measures. Following [10], Suhubi and Eringen [13] (6 January 1964) considered a linear theory and studied Rayleigh surface waves. Naghdi [14] (30 January 1964) used couple stresses in elasticity to motivate a thoughtful examination of shell theory. Green and Rivlin [15] (18 March 1964) extended [12] by following Truesdell and Toupin [18] in the use of generalized velocities, body and surface forces and stresses, and obtained sufficient conditions under which the equations of motion and surface conditions of [18] hold. Toupin [16] (1 June 1964) formulated a general nonlinear theory of elasticity with microstructure, gave an admirable survey of existing literature and showed how the nonlinear strain measures reduce to those utilized in Mindlin's [8] linear theory. Green *et al.* [17] (10 June 1964) gave a new definition of multipolar displacements, identified the place of directors in the theory of [15], and presented a complete thermodynamical theory of elastic media with directors.

Although all of these works provide insight, our formulation of the linear theory of thermoelasticity with microstructure will be guided mainly by the complementary works of Mindlin [8] and Toupin [16], both of whom obtained the equations of motion and the twelve traction boundary conditions by use of Hamilton's principle. The governing equations of this theory can also be obtained, following Green and Rivlin [12], by postulating an energy balance equation subject to certain invariance requirements. This approach was taken by Fox [19], who developed a continuum theory of dislocations for elastic materials with microstructure, and by Allen *et al.* [20] in a study of fluids with deformable microstructure.

We preface consideration of representative variational principles formulated within the context of theories of generalized continua and materials with microstructure with some remarks on kinematics. Toupin's approach to the kinematics of media with microstructure [16], which we adopt, involves associating a triad of directors with each material particle of the medium. In general, the directors are allowed to deform and rotate independently of the deformation field associated with material particles. In this event a deformed configuration of the medium is defined by twelve mapping functions: three for material particle displacements, nine for microstructure (director) deformations. The nine microstructure deformation mapping functions can be interpreted as follows: three describe the microstructure rotation field and six describe the microstructure deformation field. From [8, 20] we point out that the double force, a symmetric second order tensor, arises only when microstructure deformations are considered. If microstructure deformations are neglected and one considers only particle displacement and microstructure rotation fields, then the equations of a Cosserat continuum result [8].

Naghdi [14] introduced couple stresses and considered displacement and rotation fields in developing a variational principle in elasticity. The displacement and rotation fields were not taken to be independent in this study; the rotation was defined in the classical sense through the curl of displacement. Nowacki [21] gave a variational principle applicable to thermoelastic media with independent displacement and rotation fields. Couple stresses entered naturally in the theory and six stress equations of motion resulted: three for linear momentum balance and three for angular (spin) momentum balance.

In the present paper variational principles are given which extend those discussed above in that in addition to considering independent displacement and rotation fields we allow for independent microstructure deformations. With this kinematical structure double stress and double force tensors enter naturally and twelve stress equations of motion result: six accounting for linear and spin momentum balance and six accounting for microstructure stretch momentum balance. The stretch momentum equations enter since the microstructure is here allowed to deform as well as rotate. The present results thus complement those given earlier by Kline [22], who considered independent displacement, rotation and microstructure deformation fields, and developed a variational principle having as its Euler equations all of the field equations and boundary conditions of the linear (isothermal) theory of viscoelastic media with microstructure.

Our first variational principle assumes only that the double stress tensor and the classical strain tensor are symmetric and yields, as Euler equations: the twelve stress equations of motion and the energy balance equation (these thirteen equations are expressed in a manner so as to implicitly include all initial conditions); constitutive equations for the heat flux, and the stress, couple stress and double force tensors; the equation of state; the full set of kinematical and thermal relations (which express strain measures in terms of the displacement, rotation and microstructure deformation fields, and the thermal gradient in terms of the temperature field); the stress and couple stress boundary conditions; the particle displacement and micro-deformation boundary conditions; and the temperature and heat flux boundary conditions. By considering restrictions on the set of admissible states we also obtain a less general variational principle which is the thermoelasticity with microstructure counterpart of the principle of minimum potential energy. For additional comments and references regarding variational principles we refer the reader to [1, 5, 14, 22].

The main objective of this paper was to extend available variational principles to allow consideration of the general kinematics of thermoelastic materials with microstructure.

However, certain by-products of the work deserve mention. The role of the stretch momentum equation [the symmetric part of (2.2b)] is now abundantly clear. Rather than considering separately stretch and spin momentum equations, one instead regards (2.2b) as providing nine microstructure momentum balance equations and interprets the symmetry of the double force tensor as merely placing restrictions on the form of its constitutive equation (in complete analogy with the treatment of Cauchy stress in nonpolar mechanics), see (2.4b), (2.5). Also, we observe that by appropriately regarding the specific entropy as the fundamental thermodynamic variable it is possible to explicitly state (Section 5) how the equation of state and the energy equation should be handled in variational formulations of other theories of generalized elastic continua. Further, this work shed light on the role of the microstructure inertia tensor **I** [which appears in (2.2b)]. It was found necessary to require **I** to be nonsingular *if* one wanted to allow specification of arbitrary (in the context of a linear theory) initial conditions on micro-deformations and micro-deformation velocities, i.e. initial conditions on the microstructure motion which are totally unrelated to initial conditions on material particle displacements and velocities, see (2.6). For nonpolar materials (classical continua) the microstructure motion, as it were, is completely determined by the material particle motion [16], and thus independent initial conditions cannot be prescribed. Consistent with this, to reduce microstructure theory to the classical theory it is necessary to require **I** to vanish (see Section 5). Thus the variational formulation implies **I** is a generalized density. Interestingly, Dahler and Scriven [23] concluded **I** was nonsingular based on statistical physics considerations.

In closing this section we acknowledge our broad debt to the literature.

2. THE INITIAL-BOUNDARY VALUE PROBLEM

In this section we record the full system of equations for linear coupled thermoelasticity for (inhomogeneous and anisotropic) materials with microstructure. Let $\bar{V}$ be the region of space occupied by the medium, where $\bar{V}$ is the closure of an open, bounded, connected domain of three-dimensional Euclidean space. Denote the interior of $\bar{V}$ by V and the boundary of $\bar{V}$ by S, where S is the union of a finite number of nonintersecting closed regular surfaces (in the usual sense of Kellogg [24]). Let $\mathbf{x}$ be the position vector and t the time and consider all functions of $(\mathbf{x}, t)$ as being defined on $\bar{V} \times [0, \infty)$, the Cartesian product of region $\bar{V}$ and time interval $[0, \infty)$. Functions are defined on a boundary point of $V \times (0, \infty)$ in the sense of Gurtin (see (2.3) of [1]).

We adopt the notation of [8, 20, 22] and employ a rectangular Cartesian coordinate system throughout this paper. Thus, let $\mathbf{u}(\mathbf{x}, t)$ be the displacement vector, $\boldsymbol{\psi}(\mathbf{x}, t)$ the micro-deformation tensor (the director differences tensor, see [16]), $\theta(\mathbf{x}, t)$ the temperature above a quiescent reference state T_0, $\mathbf{e}(\mathbf{x}, t)$ the classical infinitesimal strain tensor, $\boldsymbol{\gamma}(\mathbf{x}, t)$ the relative deformation tensor [8], $\boldsymbol{\kappa}(\mathbf{x}, t)$ the micro-deformation gradient tensor [8] and $\boldsymbol{\sigma}(\mathbf{x}, t)$ the thermal gradient vector.

Then the *kinematical* and *thermal* relations are

$$e_{ij} = \tfrac{1}{2}(u_{i,j}+u_{j,i}), \qquad \gamma_{ij} = u_{j,i}-\psi_{ij}, \tag{2.1a,b}$$

$$\kappa_{ijk} = \psi_{jk,i}, \qquad \sigma_i = \theta_{,i}, \quad \text{on } V\times(0, \infty). \tag{2.1c,d}$$

Let $\mathbf{t}(\mathbf{x}, t)$ be the stress tensor, $\mathbf{F}(\mathbf{x}, t)$ the body force vector, $\rho(\mathbf{x})$ the mass density, $\mathbf{m}(\mathbf{x}, t)$ the double force tensor, $\boldsymbol{\mu}(\mathbf{x}, t)$ the double stress tensor, $\mathbf{B}(\mathbf{x}, t)$ the body double force tensor (denoted by $\mathbf{\Phi}$ in [8]), $\eta(\mathbf{x}, t)$ the specific entropy per unit mass, $H(\mathbf{x}, t)$ the rate of internal heat generation per unit mass and $\mathbf{q}(\mathbf{x}, t)$ the heat flux vector. Further, $\mathbf{I}(\mathbf{x})$ is a symmetric second order tensor which represents generalized microstructure inertia coefficients [10, 13, 8].

Then the *equations of motion* and *energy* are

$$t_{ji,j}+F_i = \rho\ddot{u}_i, \tag{2.2a}$$

$$t_{ji}-m_{ji}+\mu_{kji,k}+B_{ji} = \rho I_{jk}\ddot{\psi}_{ki}, \qquad m_{ij} = m_{ji}, \tag{2.2b}$$

$$\rho\dot{\eta}T_0 = \rho H - q_{i,i} \quad \text{on } V\times(0,\infty). \tag{2.2c}$$

The antisymmetric part of (2.2b) gives the balance of spin momentum, the symmetric part the balance of stretch momentum. We note from [8, 20] that Mindlin's τ is our $\mathbf{m}$, his $\boldsymbol{\sigma}$ our $\mathbf{t}-\mathbf{m}$.

We now recall that in terms of the Helmholtz free energy function A the linear theory of thermoelasticity with microstructure requires [8, 19, 17, 10, 25]

$$\begin{aligned} t_{kj}-m_{kj} &= \rho\frac{\partial A}{\partial\gamma_{kj}}, & m_{kj} &= \rho\frac{\partial A}{\partial e_{kj}}, \\ \mu_{kji} &= \rho\frac{\partial A}{\partial\kappa_{kji}}, & \eta &= -\frac{\partial A}{\partial\theta}. \end{aligned} \tag{2.3}$$

Then, in view of (2.3) and guided by Mindlin's expression for the potential energy density function (equation (5.1) of [8]), the *constitutive equations* and the *equation of state* are expressed as

$$t_{kj}-m_{kj} = b_{kjmn}\gamma_{mn}+s_{kjpmn}\kappa_{pmn}+g_{kjmn}e_{mn}-p_{kj}\theta, \tag{2.4a}$$

$$m_{kj} = c_{kjmn}e_{mn}+f_{kjpmn}\kappa_{pmn}+g_{mnkj}\gamma_{mn}-h_{kj}\theta, \tag{2.4b}$$

$$\mu_{pmn} = a_{ijkpmn}\kappa_{ijk}+s_{kjpmn}\gamma_{kj}+f_{kjpmn}e_{kj}-r_{pmn}\theta, \tag{2.4c}$$

$$q_i = -k_{ij}\sigma_j, \tag{2.4d}$$

$$\rho C\theta = T_0[\rho\eta-h_{kj}e_{kj}-p_{kj}\gamma_{kj}-r_{pmn}\kappa_{pmn}] \quad \text{on } V\times(0,\infty). \tag{2.4e}$$

In (2.4) the quantities $\mathbf{b}, \mathbf{s}, \mathbf{g}, \mathbf{p}, \mathbf{c}, \mathbf{f}, \mathbf{h}, \mathbf{a}, \mathbf{r}, C$ and $\mathbf{k}$ are material coefficients which may be functions of position vector $\mathbf{x}$ and which satisfy the symmetry relations

$$\begin{aligned} &b_{kjmn} = b_{mnkj}, \qquad g_{kjmn} = g_{kjnm}, \\ &c_{kjmn} = c_{kjnm} = c_{mnkj}, \\ &f_{kjpmn} = f_{jkpmn}, \qquad h_{kj} = h_{jk}, \\ &a_{ijkpmn} = a_{pmnijk}, \qquad k_{ij} = k_{ji} \quad \text{on } V. \end{aligned} \tag{2.5}$$

Associated with the system of field equations (2.1), (2.2), (2.4), (2.5) are the *initial conditions*

$$u_i(\mathbf{x},0) = d_i(\mathbf{x}), \qquad \dot{u}_i(\mathbf{x},0) = v_i(\mathbf{x}), \tag{2.6a}$$

$$\psi_{ij}(\mathbf{x},0) = \hat{d}_{ij}(\mathbf{x}), \qquad \dot{\psi}_{ij}(\mathbf{x},0) = \hat{v}_{ij}(\mathbf{x}), \tag{2.6b}$$

$$\theta(\mathbf{x},0) = \theta_0(\mathbf{x}) \quad \text{on } \bar{V}, \tag{2.6c}$$

where $\mathbf{d}, \mathbf{v}, \hat{\mathbf{d}}, \hat{\mathbf{v}}$ and θ_0 are prescribed functions accounting for the initial displacements, velocities, micro-deformations, micro-deformation velocities and temperature distribution, respectively.

Preparatory to stating boundary conditions we let $\mathbf{n}$ denote the unit outward normal to boundary S and term a point $\mathbf{x} \in S$ a regular point if $\mathbf{n}$ is continuous at $\mathbf{x}$ [1]. Further, (S_t, S_u), (S_μ, S_ψ) and (S_Q, S_θ) are introduced as a system of complementary regular subsets of S with, for example, $\bar{S}_t$ denoting the closure of S_t. Then we adjoin to the field equations and initial conditions the *mixed boundary conditions*

$$t_i = t_{ji}n_j = \hat{t}_i \text{ on } S_t \times [0, \infty), \tag{2.7a}$$

$$u_i = \hat{u}_i \text{ on } S_u \times [0, \infty), \tag{2.7b}$$

$$\mu_{ji} = \mu_{kji}n_k = \hat{\mu}_{ji} \text{ on } S_\mu \times [0, \infty), \tag{2.7c}$$

$$\psi_{ij} = \hat{\psi}_{ij} \text{ on } S_\psi \times [0, \infty), \tag{2.7d}$$

$$Q = q_i n_i = \hat{Q} \text{ on } S_Q \times [0, \infty), \tag{2.7e}$$

$$\theta = \hat{\theta} \text{ on } S_\theta \times [0, \infty). \tag{2.7f}$$

Here $\hat{\mathbf{t}}, \hat{\mathbf{u}}, \hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\psi}}, \hat{Q}$ and $\hat{\theta}$ are the given surface stress tractions, displacements, double tractions, micro-deformations, normal heat flux and temperature, respectively.

3. ALTERNATIVE FORMULATIONS

From the preceding section we observe that the mixed problem consists of finding a set of functions $[\mathbf{u}, \boldsymbol{\psi}, \theta, \mathbf{e}, \boldsymbol{\gamma}, \boldsymbol{\kappa}, \boldsymbol{\sigma}, \mathbf{t}, \mathbf{m}, \boldsymbol{\mu}, \mathbf{q}, \eta]$ on $\bar{V} \times [0, \infty)$ which satisfies the field equations (2.1), (2.2), (2.4), the initial conditions (2.6) and the boundary conditions (2.7). It is now necessary to specify certain regularity assumptions; following Gurtin [1] we adopt as *hypotheses on the data*:

(i) $\rho > 0$ and $I_{ij} = I_{ji}$ are continuously differentiable and $\det \mathbf{I} > 0$ on $\bar{V}$;
(ii) $\mathbf{b}, \mathbf{s}, \mathbf{g}, \mathbf{p}, \mathbf{c}, \mathbf{f}, \mathbf{h}, \mathbf{a}, \mathbf{r}, C > 0$ and $\mathbf{k}$ are continuously differentiable on $\bar{V}$ and satisfy (2.5);
(iii) $\mathbf{d}$ and $\hat{\mathbf{d}}$ are continuously differentiable on $\bar{S}$, and $\mathbf{v}, \hat{\mathbf{v}}$ and θ_0 are continuous on $\bar{S}$;
(iv) $\mathbf{F}, \mathbf{B}, H$ are continuous on $\bar{V} \times [0, \infty)$;
(v) $\hat{\mathbf{u}}, \hat{\boldsymbol{\psi}}$ and $\hat{\theta}$ are continuous on $\bar{S}_u \times [0, \infty)$, $\bar{S}_\psi \times [0, \infty)$ and $\bar{S}_\theta \times [0, \infty)$, respectively;
(vi) $\hat{\mathbf{t}}, \hat{\mathbf{u}}$ and $\hat{Q}$ are piecewise regular (see [1], p. 35) on $\bar{S}_t \times [0, \infty)$, $\bar{S}_\mu \times [0, \infty)$ and $\bar{S}_Q \times [0, \infty)$, respectively.

To characterize the solution of the foregoing mixed problem by means of variational principles it is convenient to first introduce an admissible state. Considering functions of $(\mathbf{x}, t)$ we shall use the standard definition for the function class $C^{M,N}$ on $\bar{V} \times [0, \infty)$, recalling that the first index indicates the order of spatial differentiation while the second refers to time differentiation ([1], p. 35). Then by an *admissible state* we mean the ordered array of functions

$$\mathscr{A} = [\mathbf{u}, \boldsymbol{\psi}, \theta, \mathbf{e}, \boldsymbol{\gamma}, \boldsymbol{\kappa}, \boldsymbol{\sigma}, \mathbf{t}, \mathbf{m}, \boldsymbol{\mu}, \mathbf{q}, \eta]$$

defined on $\bar{V}\times[0,\infty)$ with the properties

$$
\begin{array}{llll}
\text{(a)} & u_i\in C^{1,2}, & \psi_{ij}\in C^{1,2}, & \theta\in C^{1,0}, \\
& e_{ij}\in C^{0,0}, & \gamma_{ij}\in C^{0,0}, & \kappa_{ijk}\in C^{0,0}, \\
& \sigma_i\in C^{0,0}, & t_{ij}\in C^{1,0}, & m_{ij}\in C^{0,0}, \\
& \mu_{ijk}\in C^{1,0}, & q_i\in C^{1,0}, & \eta\in C^{0,1}; \\
\text{(b)} & e_{ij}=e_{ji}, & m_{ij}=m_{ji} & \text{on} \quad \bar{V}\times[0,\infty).
\end{array}
\tag{3.1}
$$

We note in passing that one could eliminate symmetry restrictions (b) from the definition of an admissible state by following the approach of Reissner [26] in the subsequent development of variational principles.

We adopt the usual definitions ([1], p. 41) for addition of states and multiplication of a state by a scalar. Thus with α a scalar

$$
\begin{aligned}
\mathscr{A}+\tilde{\mathscr{A}} &= [\mathbf{u}+\tilde{\mathbf{u}}, \boldsymbol{\psi}+\tilde{\boldsymbol{\psi}},\dots,\eta+\tilde{\eta}], \\
\alpha\mathscr{A} &= [\alpha\mathbf{u},\alpha\boldsymbol{\psi},\dots,\alpha\eta]
\end{aligned}
\tag{3.2}
$$

and hence the set of all admissible states is a linear space.

Now a *solution of the mixed problem* is defined as an admissible state $\mathscr{A}=[\mathbf{u},\boldsymbol{\psi},\theta,\mathbf{e},\boldsymbol{\gamma},\boldsymbol{\kappa},\boldsymbol{\sigma},\mathbf{t},\mathbf{m},\boldsymbol{\mu},\mathbf{q},\eta]$ which satisfies the field equations (2.1), (2.2), (2.4), the initial conditions (2.6) and the boundary conditions (2.7). We are now in a position to discuss the

Equivalent boundary value problem

Remark 1. Let

$$\mathscr{A}=[\mathbf{u},\boldsymbol{\psi},\theta,\mathbf{e},\boldsymbol{\gamma},\boldsymbol{\kappa},\boldsymbol{\sigma},\mathbf{t},\mathbf{m},\boldsymbol{\mu},\mathbf{q},\eta]$$

be an admissible state. Then $\mathscr{A}$ is a solution of the mixed problem (of coupled thermoelasticity with microstructure) if, and only if, $\mathscr{A}$ meets the field equations (2.1), (2.4), (3.3) and boundary conditions (2.7), where

$$g*t_{ji,j}+f_i=\rho u_i, \tag{3.3a}$$

$$g*(t_{ji}-m_{ji}+\mu_{kji,k})+b_{ji}=\rho I_{jk}\psi_{ki}, \tag{3.3b}$$

$$h-g'*q_{i,i}=\rho T_0\eta \quad \text{on} \quad V\times[0,\infty). \tag{3.3c}$$

The functions g, f_i, b_{ji}, h and g' are defined as follows:

$$
\begin{aligned}
g(t) &= t, \qquad g'(t)=1 \quad (0\le t<\infty); \\
f_i(\mathbf{x},t) &= [g*F_i](\mathbf{x},t)+\rho(\mathbf{x})[tv_i(\mathbf{x})+d_i(\mathbf{x})], \\
b_{ji}(\mathbf{x},t) &= [g*B_{ji}](\mathbf{x},t)+\rho(\mathbf{x})I_{jk}(\mathbf{x})[t\hat{v}_{ki}(\mathbf{x})+\hat{d}_{ki}(\mathbf{x})], \\
h(x,t) &= [g'*H](\mathbf{x},t)+\rho(\mathbf{x})C(\mathbf{x})\theta_0(\mathbf{x}) \\
&\quad +T_0[h_{kj}(\mathbf{x})d_{j,k}(\mathbf{x})+p_{kj}(\mathbf{x})(d_{j,k}(\mathbf{x})-\hat{d}_{kj}(\mathbf{x})) \\
&\quad +r_{pmn}(\mathbf{x})\hat{d}_{mn,p}(\mathbf{x})], \qquad (\mathbf{x},t)\in\bar{V}\times[0,\infty).
\end{aligned}
\tag{3.4}
$$

Further, $a*b$ denotes the convolution of functions $a(\mathbf{x}, t)$, $b(\mathbf{x}, t)$ defined on $\bar{V}\times[0, \infty)$, and is given by

$$[a*b](\mathbf{x}, t) = \int_0^t a(\mathbf{x}, t-\tau)b(\mathbf{x}, \tau)\,d\tau \quad \text{on } \bar{V}\times[0, \infty).$$

To prove remark 1, consider an admissible state $\mathscr{A} = [\mathbf{u}, \boldsymbol{\psi}, \theta, \mathbf{e}, \boldsymbol{\gamma}, \boldsymbol{\kappa}, \boldsymbol{\sigma}, \mathbf{t}, \mathbf{m}, \boldsymbol{\mu}, \mathbf{q}, \eta]$ which by definition has the properties (3.1). Clearly we need only show that field equations (2.2) and initial conditions (2.6) are satisfied if, and only if, field equations (3.3) hold. To this end certain regularity properties in (3.1) may be relaxed; for example, it suffices to consider $u_i \in C^{0,2}$, $\psi_{ij} \in C^{0,2}$. Gurtin [1] proved field equations (2.2a) and initial conditions (2.6a) are satisfied if, and only if, field equations (3.3a) are satisfied. We now prove field equations (2.2b) and initial conditions (2.6b) hold if, and only if, field equations (3.3b) are satisfied.

Say (3.3b) holds and use (3.4) to obtain

$$\rho I_{jk}(\psi_{ki} - t\hat{v}_{ki} - \hat{d}_{ki}) = g*(t_{ji} - m_{ji} + \mu_{kji,k} + B_{ji}). \tag{3.5}$$

Since by assumption $\mathbf{I}$ is nonsingular introduce $\mathbf{I}^{-1}$, the inverse to $\mathbf{I}$, and write (3.5) as

$$\rho(\psi_{mi} - t\hat{v}_{mi} - \hat{d}_{mi}) = I^{-1}_{mj} g*(t_{ji} - m_{ji} + \mu_{kji,k} + B_{ji}). \tag{3.6}$$

Using the definition of g, (3.4), it follows from (3.6) that $\psi_{mi}(\mathbf{x}, 0) = \hat{d}_{mi}(\mathbf{x})$, $\dot{\psi}_{mi}(\mathbf{x}, 0) = \hat{v}_{mi}(\mathbf{x})$. With initial conditions (2.6b) established use (2.6b), (3.5) and integration by parts to find

$$0 = g*(t_{ji} - m_{ji} + \mu_{kji,k} + B_{ji} - \rho I_{jk}\ddot{\psi}_{ki}). \tag{3.7}$$

From (3.7) and the Titchmarsh theorem [3] follows field equations (2.2b). Conversely, assume (2.2b), (2.6b) hold. Then from (3.7) and by reversing the above argument obtain (3.3b).

Finally, to complete the proof of remark 1 we are to show equations (2.2c), (2.6c) are equivalent to (3.3c). This can be accomplished following Nickell and Sackman [5], however, one comment is in order; it is not necessary to assume the equation of state (2.4e) holds for $t = 0$, rather, this fact follows naturally. For example, let $\mathscr{A}$ be an admissible state which meets (2.1), (2.2), (2.4), (2.6), (2.7). To show (3.3c) holds observe that from (2.1a,b,c) equation of state (2.4e) may be expressed in terms of $\theta, \eta, \mathbf{u}, \boldsymbol{\psi}$ on $V\times(0, \infty)$. From regularity properties (3.1) and the fact that $\theta, \mathbf{u}, \boldsymbol{\psi}$ take on finite, specified values [through (2.6)] over $\bar{V}$ at $t = 0$, it follows that $\eta(\mathbf{x}, 0)$ may be expressed in terms of $\theta_0, \mathbf{d}, \hat{\mathbf{d}}$ over V. The validity of (3.3c) then follows from the proof of Theorem 4.2 of [5] if one uses the definition of h from (3.4). Conversely, to show (2.2c), (2.6c) hold if $\mathscr{A}$ is an admissible state which meets (2.1), (2.4), (3.3), (2.7) we again use the fact that from (2.1), (2.4e) θ may be expressed in terms of $\eta, \mathbf{u}, \boldsymbol{\psi}$ on $V\times(0, \infty)$. Now by (3.3c) $\eta(\mathbf{x}, 0)$ is well defined on V. Thus by (3.1), (2.2a,b), (2.4e) $\theta(\mathbf{x}, 0)$ may be expressed in terms of $\eta(\mathbf{x}, 0)$, $\mathbf{d}(\mathbf{x})$, $\hat{\mathbf{d}}(\mathbf{x})$. From this, (3.3c) and the definition of h in (3.4) it is easy to establish (2.2c), (2.6c). The argument is analogous to that given above pertaining to (3.5); also, see [5].

We point out that the expressions given for field equations (3.3) could be developed by use of the Laplace transform [4, 5]. Also note that the full set of initial conditions enter (3.3) through the definitions (3.4).

Guided by remark 1 other alternative formulations can be made. To give an example we now define a *deformation and temperature field corresponding to a solution of the mixed problem* as an ordered array of functions $[\mathbf{u}, \boldsymbol{\psi}, \theta]$ such that there exist functions $\mathbf{e}, \boldsymbol{\gamma}, \boldsymbol{\kappa}, \boldsymbol{\sigma}$,

t, **m**, **μ**, **q**, η with the property that [**u**, **ψ**, θ, **e**, **γ**, **κ**, **σ**, **t**, **m**, **μ**, **q**, η] is a solution to the mixed problem. Then, in analogy to Theorems 3.3 of [1] and 4.4 of [5], we state

Remark 2. Let $u_i \in C^{2,2}$, $\psi_{ji} \in C^{2,2}$ and $\theta \in C^{2,1}$. Then [**u**, **ψ**, θ] is a deformation and temperature field corresponding to a solution of the mixed problem of coupled thermoelasticity with microstructure if, and only if,

$$\begin{aligned} g^*[&(b_{kjmn}+g_{mnkj})(u_{n,m}-\psi_{mn}) \\ &+(g_{kjmn}+c_{kjmn})u_{m,n}+(s_{kjpmn}+f_{kjpmn})\psi_{mn,p} \\ &-(p_{kj}+h_{kj})\theta]_{,k}+f_j = \rho u_j, \end{aligned} \tag{3.8a}$$

$$\begin{aligned} g^*\{&b_{kjmn}(u_{n,m}-\psi_{mn})+s_{kjpmn}\psi_{mn,p} \\ &+g_{kjmn}u_{m,n}-p_{kj}\theta+[a_{pmnikj}\psi_{mn,p} \\ &+s_{mnikj}(u_{n,m}-\psi_{mn})+f_{mnikj}u_{m,n}-r_{ikj}\theta]_{,i}\} \\ &+b_{kj} = \rho I_{km}\psi_{mj}, \end{aligned} \tag{3.8b}$$

$$\begin{aligned} h+g'^*(k_{ij}\theta_{,j})_{,i} &= \rho C\theta+T_0[h_{kj}u_{j,k}+p_{kj}(u_{j,k}-\psi_{kj}) \\ &+r_{pmn}\psi_{mn,p}] \quad \text{on } V\times[0,\infty); \end{aligned} \tag{3.8c}$$

and

$$\begin{aligned} [&(b_{kjmn}+g_{mnkj})(u_{n,m}-\psi_{mn})+(g_{kjmn}+c_{kjmn})u_{m,n} \\ &+(s_{kjpmn}+f_{kjpmn})\psi_{mn,p}-(p_{kj}+h_{kj})\theta]n_k = \hat{t}_j \quad \text{on } S_t\times[0,\infty), \end{aligned} \tag{3.9a}$$

$$u_i = \hat{u}_i \quad \text{on } S_u\times[0,\infty), \tag{3.9b}$$

$$\begin{aligned} [&a_{pmnikj}\psi_{mn,p}+s_{mnikj}(u_{n,m}-\psi_{mn})+f_{mnikj}u_{m,n} \\ &-r_{ikj}\theta]n_i = \hat{\mu}_{kj} \quad \text{on } S_\mu\times[0,\infty), \end{aligned} \tag{3.9c}$$

$$\psi_{kj} = \hat{\psi}_{kj} \quad \text{on } S_\psi\times[0,\infty), \tag{3.9d}$$

$$-k_{ij}\theta_{,j}n_i = \hat{Q} \quad \text{on } S_Q\times[0,\infty), \tag{3.9e}$$

$$\theta = \hat{\theta} \quad \text{on } S_\theta\times[0,\infty). \tag{3.9f}$$

To prove remark 2 say **u**, **ψ**, θ meet (3.8), (3.9). Define **e**, **γ**, **κ**, **σ** through (2.1), and **t**, **m**, **μ**, **q**, η by (2.4). Then use (2.5), (3.9) to show boundary conditions (2.7) hold. With (2.1), (2.4), (2.5) equations (3.8) imply (3.3) and thus remark 1 yields that [**u**, **ψ**, θ, **e**, **γ**, **κ**, **σ**, **t**, **m**, **μ**, **q**, η] is a solution to the mixed problem. Conversely (2.1), (2.4), (2.5), (2.7), (3.3) imply (3.8), (3.9) to complete the proof.

4. VARIATIONAL PRINCIPLES FOR THE MIXED PROBLEM

In this section we give two variational principles characterizing coupled thermoelasticity with microstructure. The first principle is more general since it treats admissible states which are not required to meet any of the field equations, initial conditions or boundary conditions.

The term functional is used to identify a real valued function whose domain is a subset of a linear space (recall the set of all admissible states is a linear space). If L is a linear space,

K a subset of L and $\Omega\{.\}$ a functional defined on K, let

$$\mathscr{A}, \tilde{\mathscr{A}} \in L, \qquad \mathscr{A} + \alpha\tilde{\mathscr{A}} \in K \text{ for every real } \alpha \tag{4.1}$$

and formally define the notation

$$\delta_{\tilde{\mathscr{A}}}\Omega\{\mathscr{A}\} = \frac{\mathrm{d}}{\mathrm{d}\alpha}\Omega\{\mathscr{A} + \alpha\tilde{\mathscr{A}}\}_{\alpha=0} \tag{4.2}$$

The variation of $\Omega\{.\}$ is zero at $\mathscr{A}$ over K and is written

$$\delta\Omega\{\mathscr{A}\} = 0 \text{ over } K, \tag{4.3}$$

if, and only if, $\delta_{\tilde{\mathscr{A}}}\Omega\{\mathscr{A}\}$ exists and equals zero for every choice of $\tilde{\mathscr{A}}$ consistent with (4.1).

Recall that $g(t) = t$, $g'(t) = 1$ $(0 \leq t < \infty)$ and functions f_i, b_{ji} and h are defined on $\overline{V} \times [0, \infty)$ through (3.4) and include the prescribed initial values for displacements, velocities, micro-deformations, micro-deformation velocities and temperature.

First variational principle

Let K be the set of all admissible states. Let $\mathscr{A} = [\mathbf{u}, \boldsymbol{\psi}, \theta, \mathbf{e}, \boldsymbol{\gamma}, \boldsymbol{\kappa}, \boldsymbol{\sigma}, \mathbf{t}, \mathbf{m}, \boldsymbol{\mu}, \mathbf{q}, \eta] \in K$, and for each $t \in [0, \infty)$ define the functional $\Omega_t\{.\}$ on K by

$$\begin{aligned}
\Omega_t\{\mathscr{A}\} = {} & \int_{S_u} g^* t_i^* \hat{u}_i \,\mathrm{d}S + \int_{S_t} g^*(t_i - \hat{t}_i)^* u_i \,\mathrm{d}S \\
& + \int_{S_\psi} g^* \mu_{ij}{}^* \hat{\psi}_{ij} \,\mathrm{d}S + \int_{S_\mu} g^*(\mu_{ij} - \hat{\mu}_{ij})^* \psi_{ij} \,\mathrm{d}S \\
& + \int_{S_\theta} \frac{1}{T_0} g^* g'^* Q^* \hat{\theta} \,\mathrm{d}S + \int_{S_Q} \frac{1}{T_0} g^* g'^* (Q - \hat{Q})^* \theta \,\mathrm{d}S \\
& - \int_V \Big\{ (g^* t_{ji,j} + f_i - \tfrac{1}{2}\rho u_i)^* u_i \\
& \qquad + (g^*(t_{ji} - m_{ji} + \mu_{kji,k}) + b_{ji} - \tfrac{1}{2}\rho I_{jk}\psi_{ki})^* \psi_{ji} \\
& \qquad + \frac{1}{T_0} g^*(g'^* q_{i,i} - h + \rho T_0 \eta)^* \theta \Big\} \,\mathrm{d}V \\
& - \int_V \Big\{ g^* \mu_{kij}{}^* \kappa_{kij} + g^*(t_{ji} - m_{ji})^* \gamma_{ji} \\
& \qquad + g^* m_{ji}{}^* e_{ji} + \frac{1}{T_0} g^* g'^* q_i{}^* \sigma_i \Big\} \,\mathrm{d}V \\
& + \int_V g^* \Big\{ \tfrac{1}{2} a_{kijpmn} \kappa_{kij}{}^* \kappa_{pmn} + \tfrac{1}{2} b_{jimn} \gamma_{ji}{}^* \gamma_{mn} \\
& \qquad + \tfrac{1}{2} C_{jimn} e_{ji}{}^* e_{mn} + s_{kjpmn} \gamma_{kj}{}^* \kappa_{pmn} \\
& \qquad + f_{kjpmn} e_{kj}{}^* \kappa_{pmn} + g_{kjmn} \gamma_{kj}{}^* e_{mn} \\
& \qquad - \frac{1}{2T_0} g'^* k_{ij} \sigma_i{}^* \sigma_j + \frac{T_0}{2\rho C}(\rho\eta - h_{ji} e_{ji} \\
& \qquad - p_{ji}\gamma_{ji} - r_{kij}\kappa_{kij})^* (\rho\eta - h_{mn} e_{mn} \\
& \qquad - p_{mn}\gamma_{mn} - r_{pmn}\kappa_{pmn}) \Big\} \,\mathrm{d}V.
\end{aligned} \tag{4.4}$$

Then

$$\delta\Omega_t\{\mathscr{A}\} = 0 \qquad (0 \le t < \infty) \tag{4.5}$$

if and only if $\mathscr{A}$ is a solution of the mixed problem.

Proof. Let $\tilde{\mathscr{A}} = [\tilde{\mathbf{u}}, \tilde{\boldsymbol{\psi}}, \tilde{\theta}, \tilde{\mathbf{e}}, \tilde{\boldsymbol{\gamma}}, \tilde{\boldsymbol{\kappa}}, \tilde{\boldsymbol{\sigma}}, \tilde{\mathbf{t}}, \tilde{\mathbf{m}}, \tilde{\boldsymbol{\mu}}, \tilde{\mathbf{q}}, \tilde{\eta}] \in K$, from which it follows that $\mathscr{A} + \alpha\tilde{\mathscr{A}} \in K$ for every scalar α. Then by (4.4), (4.2), (2.5), property (b) of admissible states, properties of the convolution [3], and the divergence theorem

$$\begin{aligned}
\delta_{\tilde{\mathscr{A}}}\Omega_t\{\mathscr{A}\} = {} & \int_{S_u} g*(\hat{u}_i - u_i)*\tilde{t}_i \,\mathrm{d}S + \int_{S_t} g*(t_i - \hat{t}_i)*\tilde{u}_i \,\mathrm{d}S \\
& + \int_{S_\psi} g*(\hat{\psi}_{ij} - \psi_{ij})*\tilde{\mu}_{ij}\,\mathrm{d}S + \int_{S_\mu} g*(\mu_{ij} - \hat{\mu}_{ij})*\tilde{\psi}_{ij}\,\mathrm{d}S \\
& + \int_{S_\theta} \frac{1}{T_0} g*g'*(\hat{\theta} - \theta)*\tilde{Q}\,\mathrm{d}S + \int_{S_Q} \frac{1}{T_0} g*g'*(Q - \hat{Q})*\tilde{\theta}\,\mathrm{d}S \\
& - \int_V \Big\{ (g*t_{ji,j} + f_i - \rho u_i)*\tilde{u}_i \\
& \qquad + [g*(t_{ji} - m_{ji} + \mu_{kji,k}) + b_{ji} - \rho I_{jk}\psi_{ki}]*\tilde{\psi}_{ji} \\
& \qquad + \frac{1}{T_0} g*(g'*q_{i,i} - h + \rho T_0 \eta)*\tilde{\theta} \Big\}\,\mathrm{d}V \\
& + \int_V \Big\{ g*(u_{i,j} - \psi_{ji} - \gamma_{ji})*(\tilde{t}_{ji} - \tilde{m}_{ji}) \\
& \qquad + g*[\tfrac{1}{2}(u_{i,j} + u_{j,i}) - e_{ji}]*\tilde{m}_{ji} + g*(\psi_{ij,k} - \kappa_{kij})*\tilde{\mu}_{kij} \\
& \qquad + \frac{1}{T_0} g*g'*(\theta_{,i} - \sigma_i)*\tilde{q}_i \Big\}\,\mathrm{d}V \\
& + \int_V g* \Big\{ \Big[-(t_{ji} - m_{ji}) + b_{jimn}\gamma_{mn} + d_{jipmn}\kappa_{pmn} \\
& \qquad + g_{jimn}e_{mn} - \frac{T_0}{\rho C} p_{ji}(\rho\eta - h_{mn}e_{mn} - p_{mn}\gamma_{mn} - r_{pmn}\kappa_{pmn}) \Big]*\tilde{\gamma}_{ji} \\
& \quad + \Big[-m_{ji} + c_{jimn}e_{mn} + f_{jipmn}\kappa_{pmn} + g_{mnji}\gamma_{mn} \\
& \qquad - \frac{T_0}{\rho C} h_{ji}(\rho\eta - h_{mn}e_{mn} - p_{mn}\gamma_{mn} - r_{pmn}\kappa_{pmn}) \Big]*\tilde{e}_{ji} \\
& \quad + \Big[-\mu_{kij} + a_{kijpmn}\kappa_{pmn} + d_{mnkij}\gamma_{mn} + f_{mnkij}e_{mn} \\
& \qquad - \frac{T_0}{\rho C} r_{kij}(\rho\eta - h_{mn}e_{mn} - p_{mn}\gamma_{mn} - r_{pmn}\kappa_{pmn}) \Big]*\tilde{\kappa}_{kij} \\
& \quad + \frac{1}{T_0} g'*[-q_i - k_{ij}\sigma_j]*\tilde{\sigma}_i \\
& \quad + \Big[-\rho\theta + \frac{T_0}{C}(\rho\eta - h_{mn}e_{mn} - p_{mn}\gamma_{mn} - r_{pmn}\kappa_{pmn}) \Big]*\tilde{\eta} \Big\}\,\mathrm{d}V.
\end{aligned} \tag{4.6}$$

Now if $\mathscr{A}$ is a solution, by remark 1, (4.6) yields

$$\delta_{\tilde{\mathscr{A}}}\Omega_t\{\mathscr{A}\} = 0 \qquad (0 \le t < \infty) \tag{4.7}$$

for every $\tilde{\mathscr{A}} \in K$ and hence implies (4.5). Conversely, to show that $\mathscr{A} \in K$ is a solution to the mixed problem whenever (4.7) holds one can merely follow the proofs given by Gurtin [1] and Nickell and Sackman [5], in that one considers various special choices for $\tilde{\mathscr{A}}$ and uses (5.6) and remark 1. We omit explicit details here noting that one will use three lemmas proved by Gurtin [1] (as well as minor extensions of these lemmas).

Following the format of Gurtin [1] and guided by remarks 1, 2 and the first variational principle it is possible to deduce a variety of other variational principles. To extend the Hellinger–Reissner principle (see [14]) to include microstructure effects one must introduce additional material coefficients so as to express constitutive relations for $\mathbf{e}, \boldsymbol{\gamma}, \boldsymbol{\kappa}, \boldsymbol{\sigma}$ in terms of $\theta, \mathbf{t}, \mathbf{m}, \boldsymbol{\mu}, \mathbf{q}$, and the equation of state in the form θ a function of $\eta, \mathbf{t}, \mathbf{m}, \boldsymbol{\mu}$. We do not record these lengthy though straightforward results here; rather, as an example, we state the principle of stationary potential energy for thermoelastic materials with microstructure.

Second variational principle

Let K be the set of all admissible states that meet the kinematical and thermal relations (2.1), the constitutive relations and equation of state (2.4), and the displacement, micro-deformation and temperature boundary conditions (2.7b, d, f). Let $\mathscr{A} = [\mathbf{u}, \boldsymbol{\psi}, \theta, \mathbf{e}, \boldsymbol{\gamma}, \boldsymbol{\kappa}, \boldsymbol{\sigma}, \mathbf{t}, \mathbf{m}, \boldsymbol{\mu}, \mathbf{q}, \eta] \in K$, and for each $t \in [0, \infty)$ define the functional $\Phi_t\{\,.\,\}$ on K by

$$\begin{aligned}\Phi_t\{\mathscr{A}\} = & -\int_{S_t} g*\hat{t}_i*u_i\,\mathrm{d}S - \int_{S_\mu} g*\hat{\mu}_{ij}*\psi_{ij}\,\mathrm{d}S - \int_{S_Q} \frac{1}{T_0}g*g'*\hat{Q}*\theta\,\mathrm{d}S \\ & + \int_V \Big\{(\tfrac{1}{2}\rho u_i - f_i)*u_i + (\tfrac{1}{2}\rho I_{jk}\psi_{ki} - b_{ji})*\psi_{ji} \\ & + \frac{1}{T_0}g*(h - \rho T_0\eta)*\theta + \tfrac{1}{2}g*\Big[\mu_{kij}*\kappa_{kij} \\ & + (t_{ji} - m_{ji})*\gamma_{ji} + m_{ji}*e_{ji} \\ & + \frac{1}{T_0}g'*q_i*\sigma_i + \theta*\rho\eta\Big]\Big\}\,\mathrm{d}V. \end{aligned} \tag{4.8}$$

Then

$$\delta\Phi_t\{\mathscr{A}\} = 0 \quad \text{over } K \quad (0 \le t < \infty) \tag{4.9}$$

if and only if $\mathscr{A}$ is a solution to the mixed problem.

Proof. Let $\tilde{\mathscr{A}} = [\tilde{\mathbf{u}}, \tilde{\boldsymbol{\psi}}, \tilde{\theta}, \tilde{\mathbf{e}}, \tilde{\boldsymbol{\gamma}}, \tilde{\boldsymbol{\kappa}}, \tilde{\boldsymbol{\sigma}}, \tilde{\mathbf{t}}, \tilde{\mathbf{m}}, \tilde{\boldsymbol{\mu}}, \tilde{\mathbf{q}}, \tilde{\eta}]$ be an admissible state and suppose

$$\mathscr{A} + \alpha\tilde{\mathscr{A}} \in K \quad \text{for every scalar } \alpha. \tag{4.10}$$

Condition (4.10) requires $\tilde{\mathscr{A}}$ to meet (2.1), (2.4) with

$$\tilde{u}_i = 0 \quad \text{on } S_u \times [0, \infty), \tag{4.11a}$$

$$\tilde{\psi}_{ij} = 0 \quad \text{on } S_\psi \times [0, \infty), \tag{4.11b}$$

$$\tilde{\theta} = 0 \quad \text{on } S_\theta \times [0, \infty). \tag{4.11c}$$

Next use (4.8), (4.2), (2.1), (2.4), (2.5), (4.11), properties of the convolution and the divergence theorem to verify that

$$\delta_{\tilde{\mathscr{A}}}\Phi_t\{\mathscr{A}\} = \int_{S_t} g*(t_i-\hat{t}_i)*\tilde{u}_i\,\mathrm{d}S + \int_{S_\mu} g*(\mu_{ij}-\hat{\mu}_{ij})*\tilde{\psi}_{ij}\,\mathrm{d}S$$

$$+\int_{S_Q}\frac{1}{T_0}g*g'*(Q-\hat{Q})*\tilde{\theta}\,\mathrm{d}S+\int_V\Big\{(\rho u_i-f_i-g*t_{ji,j})*\tilde{u}_i$$

$$+[\rho I_{jk}\psi_{ki}-b_{ji}-g*(t_{ji}-m_{ji}+\mu_{kji,k})]*\tilde{\psi}_{ji}$$

$$+\frac{1}{T_0}g*(h-\rho T_0\eta+g'*q_{i,i})*\tilde{\theta}\Big\}\,\mathrm{d}V \quad (0\le t<\infty), \tag{4.12}$$

for every $\tilde{u}_i \in C^{1,2}$, $\tilde{\psi}_{ji} \in C^{1,2}$ and $\tilde{\theta} \in C^{1,0}$ which meet (4.11). Now if $\mathscr{A}$ is a solution of the mixed problem (4.12), using remark 1, implies (4.9). Conversely, (4.12), (4.9), (2.1), (2.4), remark 1 and the fundamental lemma of the calculus of variations imply that $\mathscr{A}$ is a solution of the mixed problem to complete the proof.

5. DISCUSSION

The purpose of this section is twofold: we wish to make the variational formulation of the solution to the mixed problem of linear coupled thermoelasticity with microstructure as transparent as possible and thus indicate how it could be easily adapted to serve other theories of generalized continua, as, for example, the theory of Green *et al.* [17]; we want to briefly discuss how the theory can be reduced to one governing the behavior of a material without microstructure.

Consider the form of the functional entering the first variational principle and defined by (4.4). The surface terms require no comment; they can be easily constructed consistent with any theory of generalized continua. The first volume term involves convolutions of the basic variables $\mathbf{u}$, $\boldsymbol{\psi}$, θ with the balance equations. The key point here is that the functions necessary to describe a deformation and temperature field of the material (see remark 2) must operate (through convolutions) on their associated balance equations. Also note that an energy equation of the form (2.2c) will hold for very general theories of structured elastic media (for the present theory (2.2c) results from the linearized forms of (11), (27), (28) in [25]). Thus the counterpart, in other theories, to the first volume term in (4.4) can be easily constructed using appropriate definitions analogous to (3.4).

The second volume term in (4.4) involves convolutions of stresses, double force and heat flux with associated kinematical and thermal measures. Again, the analogy will be obvious; just note all kinematical and thermal measures appropriate to a theory, e.g. those in (2.1) for the present formulation, must enter this term.

The last volume term in (4.4) may be easily constructed from an expression for the free energy function. Note in such an expression η must enter as an independent variable rather than θ. This can be easily accomplished using the equation of state and is necessary since the operational methods leading to the equivalent boundary value problem (remark 1) force one to consider η as the fundamental thermodynamic variable. Also, dissipative terms, such as those involving the thermal gradient vector, must enter this term.

To formulate the principle of stationary potential energy observe, as a guide, that with (2.1), (2.4), (2.7b, d, f), the functional defined by (4.4) reduces to that given in (4.8).

Finally, to reduce the present theory to a theory without microstructure we note that γ, defined by (2.1b), is identically zero when the microstructure deforms and rotates along with the continuum, see [16]. In this event independent boundary and initial conditions can no longer be specified for $\boldsymbol{\psi}$. With this in mind and referring to (2.4), the present theory reduces to the classical case by requiring material coefficients **s**, **g**, **p**, **a**, **f** and **r** to vanish, along with the generalized microstructure inertia coefficients **I**. We interpret the fact certain material coefficients must vanish to mean these coefficients consist of measures of microstructure characteristic lengths, see [8].

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Абстракт—Даются варияционные принципы для динамической задачи с начальными и краевыми условиями полно сопряженной линейной термоупругости для неоднородных, анизотропных материалов с микроструктурой. Путем использования методов преобразования получаются альтернативные характеристики решения в применении к смешаным задачам. Формулируются вариационные принципы в такой форме, что ее можно сейчасже применить ко всем доступным теориям структурных и обобщенных континуумов.